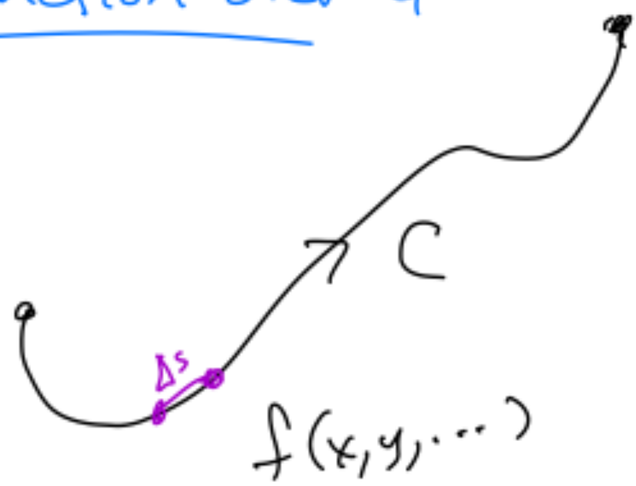
Integral of a Function over a Curve

$$\int_C f\, ds$$

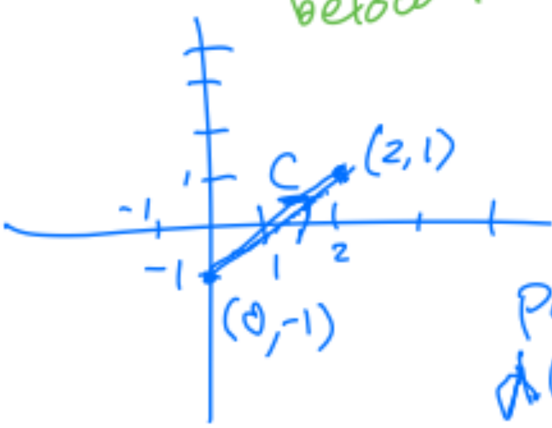


① Parametrize C

$\alpha(t)$. $\quad a \le t \le b$

② $\displaystyle\int_a^b f(\alpha(t))\, \underbrace{\|\alpha'(t)\|\, dt}_{ds}$

Example ① Find the integral of xy^2 over the curve pictured below.



$$\int_C xy^2\, ds$$

Parametrize:

$$\alpha(t) = (0,-1) + t\big((2,1)-(0,-1)\big)$$
$$= (0,-1) + t(2,2)$$
$$= (2t, -1+2t). \quad 0 \le t \le 1$$

$$ds = \|\alpha'(t)\|dt = \sqrt{2^2+2^2}\,dt$$

$$\alpha' = (2,2) \qquad = \sqrt{8}\,dt = 2\sqrt{2}\,dt$$

$$\int_C xy^2\,ds = \int_0^1 (2t)(-1+2t)^2 2\sqrt{2}\,dt$$

$$= 4\sqrt{2}\int_0^1 t\,(1-4t+4t^2)\,dt$$

$$= 4\sqrt{2}\int_0^1 t-4t^2+4t^3\,dt$$

$$= 4\sqrt{2}\left(\frac{t^2}{2}-\frac{4t^3}{3}+\frac{4t^4}{4}\right)\Big|_0^1$$

$$= 4\sqrt{2}\left(\frac{1}{2}-\frac{4}{3}+1\right) = \frac{4\sqrt{2}}{6} = \boxed{\frac{2\sqrt{2}}{3}}.$$

$$\frac{3}{2}-\frac{4}{3} = \frac{9-8}{6} = \frac{1}{6}$$

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(2) Find $\int_C f\,ds$, where $f(x,y) = xy^2$, and C is the lower half of the circle of radius 5 centered at $(0,0)$.

$$\beta(t) = (5\cos(t), 5\sin(t))$$

$$\pi \le t \le 2\pi$$

$$\beta'(t) = (-5\sin(t), 5\cos(t))$$

$$\|\beta'(t)\| = \sqrt{25\sin^2(t) + 25\cos^2(t)}$$

$$= 5\sqrt{\underbrace{\sin^2 + \cos^2}_{1}} = \boxed{5}$$

$$ds = \|\beta'(t)\|dt = 5dt.$$

$$\int_C xy^2\,ds = \int_{\pi}^{2\pi} (5\cos(t))(5\sin(t))^2(5\,dt)$$

$$= 625\int_{\pi}^{2\pi} \underbrace{\cos(t)\sin^2(t)\,dt}$$

$u = \sin(t)$ $\quad 0 \le u$

$du = \cos(t)\,dt$

$$= 625\int_0^0 u^2\,du = \boxed{0}.$$

Vector integrals ("Line integrals") over curves $\longleftrightarrow$ "integral of a differential 1-form over a curve."

$$\int_C \vec{V}\cdot\vec{ds} = \oint_C \vec{V}\cdot\vec{ds}$$

$\vec{V}$ = vector field $\vec{V} = (V_1(x,y,\dots), V_2(x,y,\dots), \dots)$

$$= \int_C V_1\,dx + V_2\,dy = \oint_C V_1\,dx + V_2\,dy.$$

What is this computing? C oriented curve (direction matters)

This $\oint_C \vec{V}\cdot \vec{ds}$ is the amount the vector field is pushing along C. $\vec{ds} = (dx, dy, \dots)$

To compute:

1. Parametrize the curve in the correct direction.
2. Plug into $V(x, y, \dots)$ and $dx, dy, \dots$

Remember: $\alpha(t) = (x(t), y(t), \dots)$

$$dx = d(x(t)) = x'(t)\,dt.$$

Example: Find the line integral of the vector field $V(x,y) = (3x, 5xy)$ along the curve that is the first quadrant of the unit circle, oriented clockwise.

$$\int_C \vec{V} \cdot \vec{ds} = \int_C V_1\,dx + V_2\,dy$$

$$= \int_C 3x\,dx + 5xy\,dy$$

Parametrize:

$\alpha(t) = (\cos(t), \sin(t))$

$0 \le t \le \frac{\pi}{2}$ wrong direction

(so parametrizes $-C$)

$\Rightarrow$

$x(t) = \cos(t)$

$dx = -\sin(t)\,dt$

$y(t) = \sin(t)$

$dy = \cos(t)\,dt$

$$\int_C 3x\,dx + 5xy\,dy = -\int_{-C} 3x\,dx + 5xy\,dy$$

$$= -\int_0^{\pi/2} 3(\cos(t))(-\sin(t)\,dt) + 5(\cos(t))(\sin(t))(\cos(t)\,dt)$$

$$= -\int_0^{\pi/2} -3\cos(t)\sin(t) + 5\sin(t)\cos^2(t)\,dt$$

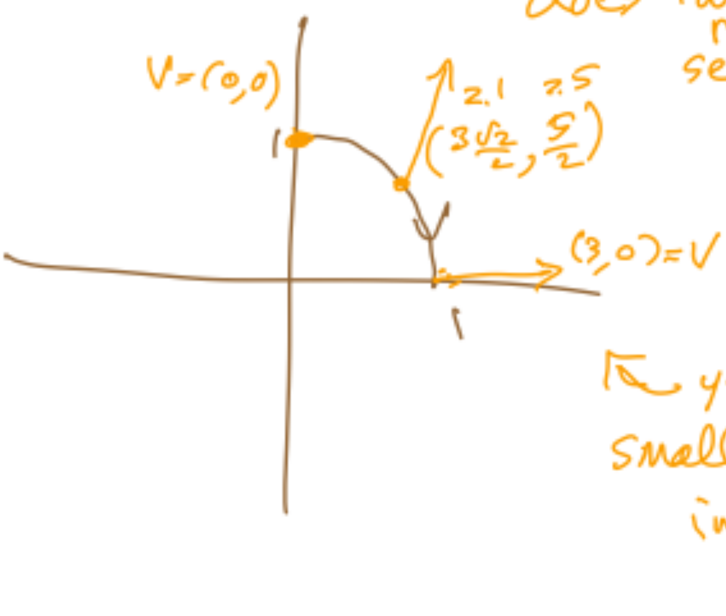
$u = \cos(t)$ $\qquad 1 \le u \le 0$

$du = -\sin(t)\,dt$

$$= -\int_{u=1}^{0} (3u - 5u^2)\,du = \int_0^1 (3u - 5u^2)\,du$$

$$\frac{3u^2}{2} - \frac{5u^3}{3}\Big|_0^1 = \frac{3}{2} - \frac{5}{3} = \frac{9-10}{6} = \boxed{-\frac{1}{6}}.$$

does this make sense with picture?



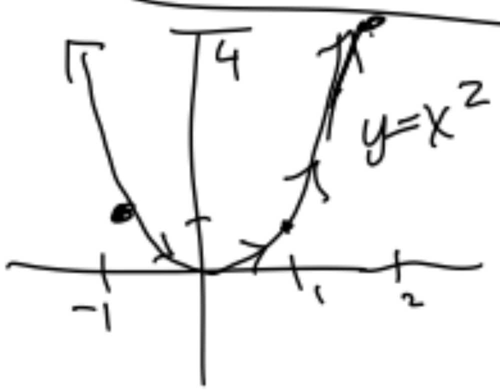
$V = (3x, 5xy)$

yes, should be small negative line integral.

Example Find

$$\oint_C 3x^2y\,dx + x^3dy,$$

where C is the part of the parabola $y=x^2$ from $x=-1$ to $x=2$.



$$\alpha(t) = (\underbrace{t}_{x(t)}, \underbrace{t^2}_{y(t)})$$

$$-1 \le t \le 2$$

$$\alpha'(t) = (\underbrace{1}_{x'(t)}, \underbrace{2t}_{y'(t)})$$

$$dx = x'(t)dt = 1\,dt$$
$$dy = y'(t)dt = 2t\,dt$$

$$\oint_C 3x^2y\,dx + x^3dy = \int_{t=-1}^{2} 3(t)^2(t^2)dt + (t)^3(2t\,dt)$$

$$= \int_{-1}^{2} (3t^4 + 2t^4)dt = \int_{-1}^{2} 5t^4 dt$$

$$= t^5\Big|_{-1}^{2} = 2^5 - (-1)^5 = 32-(-1)$$
$$= \boxed{33}.$$

Fundamental Thm of Calculus for Line Integrals (FTCLI)

Remember FTC: $\int_a^b F'(x)\,dx = F(b) - F(a)$

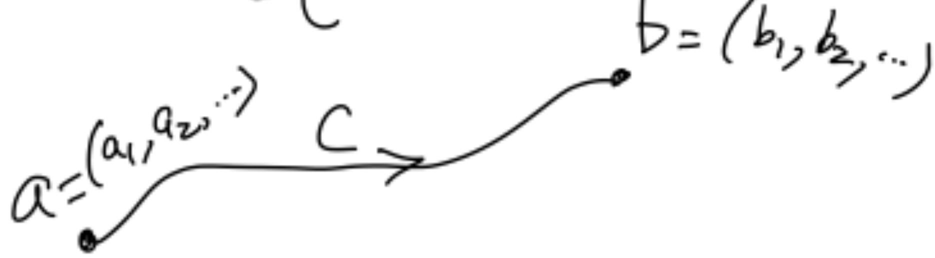
$$\int_a^b d(F) = F(b) - F(a).$$

For function $g(x, y, \ldots)$, the differential dg is

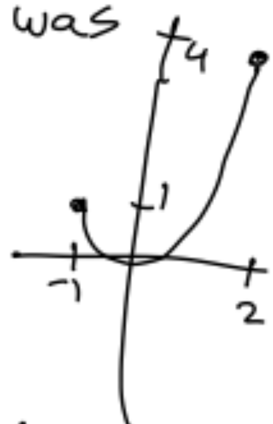
$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy + \ldots$$

FTCLI

$$\int_C dg = g(b) - g(a)$$



Example The last integral was



$$\oint_C 3x^2y\,dx + x^3\,dy$$

Notice: $d(x^3y) = \frac{\partial}{\partial x}(x^3y)\,dx + \frac{\partial}{\partial y}(x^3y)\,dy$

$$= 3x^2y\,dx + x^3\,dy.$$

Thus,

$$\oint_C 3x^2y\,dx + x^3\,dy = \oint_C d(x^3y) = x^3y \Big|_{a=(-1,1)}^{b=(2,4)}$$

$$= (2^3\cdot 4) - (-1)^3(1) = 32 - (-1) = 33 \checkmark$$

Proof of FTC LI:

$$\oint_C dg = \oint_C g_x\,dx + g_y\,dy + \cdots$$

parametrize $\alpha(t) = (x(t), y(t), \ldots)$ $t_0 \le t \le t_1$

$\alpha'(t) = (x'(t), y'(t), \ldots)$

$$= \int_{t_0}^{t_1} g_x(x(t), y(t), \ldots)\,x'(t)\,dt + g_y(x(t), y(t), \ldots)\,y'(t)\,dt + \cdots$$

$$= \int_{t_0}^{t_1} \underbrace{g'(\alpha(t)) \cdot \alpha'(t)}\, dt$$

$$= \int_{t_0}^{t_1} \frac{d}{dt}\Big(g(\alpha(t))\Big)\, dt \overset{FTC}{=} g(\alpha(t)) \Big|_{t_0}^{t_1}$$

$$= g(\alpha(t_1)) - g(\alpha(t_0))$$

$$= g(b) - g(a). \quad \blacksquare$$

If $V = (V_1, V_2, V_3, \ldots)$ is a vector field.

$\longleftrightarrow$ differential form $V_1\,dx + V_2\,dy + \cdots,$

If $dg = V_1\,dx + V_2\,dy + \cdots$

then $V_1 = g_x$, $V_2 = g_y, \ldots$

ie. $V = \nabla g$

(Vector form of FTCLI) $\oint_C \vec{\nabla} g \cdot \vec{ds} = g(b) - g(a).$

In the last example,

$$V = (3x^2y, x^3) = \nabla(x^3y),$$

We say that the vector field V is <u>conservative</u> if $V = \nabla g$ for some function g.

In differential form language,

$$\underbrace{V_1\,dx + V_2\,dy + \cdots}_{} = dg$$

is called an <u>exact</u> differential form.

How can we tell if a vector field is conservative?

$$V = (V_1, V_2, \ldots)$$

(1) Try to solve

$$g_x = V_1$$
$$g_y = V_2$$
$$\vdots$$

(2) This is true that you

can find a g if and only if

$$d(V_1\,dx + V_2\,dy + \dots) = 0.$$

<u>example:</u> $V = (3x^2y, x^3)$

1-form $\quad 3x^2y\,dx + x^3dy = \omega$

$$d\omega = d(3x^2y\,dx + x^3dy)$$

$$= d(3x^2y)\wedge dx + d(x^3)\wedge dy$$

$dx\wedge dx = 0$
$dx\wedge dy = -dy\wedge dx$

$$= [6xy\,dx + 3x^2dy]\wedge dx + [3x^2dx + 0\,dy]\wedge dy$$

$$= 6xy\,\underbrace{dx\wedge dx}_{0} + 3x^2dy\wedge dx + 3x^2dx\wedge dy + \underbrace{0\,dy\wedge dy}_{0}$$

$$= 3x^2dy\wedge dx + 3x^2dx\wedge dy$$

$$= -3x^2dx\wedge dy + 3x^2dx\wedge dy$$

$$= 0. \checkmark$$

This tells us that $V = (3x^2y, x^3)$ is <u>conservative</u>.

Consequence.

$$\oint_C dg = \oint_D dg = g(b) - g(a)$$

If the vector field is conservative, its line integrals only depend on the end points. "independent of path"